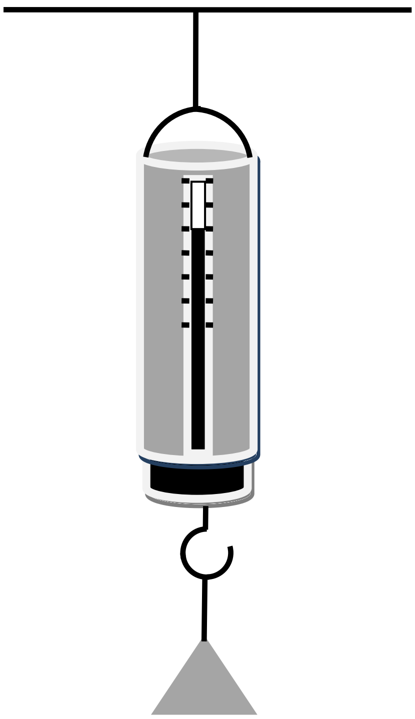
Variable Forces and Differential Equations



A spring balance measures the weight for a range of items by exerting an equal and opposite force to the gravitational force acting on a mass attached to the hook. The spring balance is therefore capable of applying a variable force, the source of which is the material properties of a spring. When in equilibrium, the spring balance and mass attached to the hook causes the spring to extend from an initial position until the resultant force is zero. Provided the structure of the spring is unaltered by these forces, the tension T in the spring is proportional to the extension x of the spring from the natural length of the spring.

The tension due to the spring is an example of a force which is a function of displacement:

$$T = f(x)$$

Hooke's Law, empirically determined (determined by experiment), states for a spring of natural length $l\ m$ when extended $x\ m$ beyond the natural length exerts a tension T proportional to the extension x. Introducing the constant $\lambda\ N$ known as modulus of elasticity for a particular spring (or extensible string), the tension due to the extension of the spring is given by:

$$T = \frac{\lambda}{l}x$$

The term natural length means the length of a spring before any external forces act to stretch or compress the spring.

If a particle is attached to a light spring and the spring is stretched to produce a displacement $x\ m$ from the natural length of the spring, then the force acting upon the particle due to the spring is given by

$$F = -\frac{\lambda}{l}x$$

Applying Newton's second law of motion $F = ma$, where $a = \frac{d^2x}{dt^2}$ the equation can be written in terms of x and derivatives of x as follows.

$$m\frac{d^2x}{dt^2} = -\frac{\lambda}{l}x \quad \cdots \quad (1)$$

Equation (1) is a second order linear differential equation, the solution of which provides the displacement as a function of time t s in the form $x = x(t)$. Differential equations are often encountered when studying dynamics, therefore before returning to problems relating to the motion of particles attached to elastic strings and springs the technical aspects of differential equations will be considered.

Differential Equations

Ordinary differential equations involve a function and derivative of the function with respect to an independent variable. For example the displacement from an origin of a particle travelling in a straight line might be expressed in the form of a differential equation for the displacement $x = x(t)$ in the form

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2e^{-t} \quad \cdots \quad (2)$$

A differential equation is a prescription for how a function $x(t)$ and functions obtained by differentiating the function can be combined to produce a specific function, in this case $2e^{-t}$.

Whenever the derivative of a function is involved, a certain amount of information is lost. The integral of a derivative of a function is the function plus an arbitrary constant. The arbitrary constant represents the lost information resulting from when the derivative is calculated. For example,

$$\frac{d[x^2]}{dx} = 2x \; and \; \frac{d[x^2 + 3]}{dx} = 2x$$

The two functions x^2 and $x^2 + 3$ both have the same derivative $2x$ therefore if presented with the derivative alone, the precise nature of the function is unknown; hence the use of a constant of integration whenever a function is integrated.

$$\int 2x \, dx = x^2 + C$$

Combining derivatives to form a differential equation for a function also means information about the function is missing within the definition and for this reason the solution to a differential equation must be expressed as a family of solutions corresponding to constants introduced to accommodate the potential loss of information associated with the derivatives. A general solution to Equation (2) is

$$x(t) = Ae^{-3t} + Be^{-2t} + e^{-t}$$

A and B are constants yet to be determined. Both $x(t) = 2e^{-3t} + 10e^{-2t} + e^{-t}$ and $x(t) = 3e^{-3t} + 3e^{-2t} + e^{-t}$ are solutions to the differential equation as are any number of other choices for the values of A and B. For a given problem, if at a given time the position and the derivative of position are known, then a specific solution from the set of solutions represented by Equation (3) can be obtained. The method used to establish solutions to equations of the standard form, of which Equation (2) is an example, will be discussed in detail later.

Solving general differential equations is a large subject, so for sixth form mechanics the types of differential equations considered are limited to a subset of equations which fit standard forms. Equations (1) and (2) are linear second order differential equations with constant coefficients. To begin with, solutions for certain standard forms of first order differential equations will be considered.

The differential equations used to model the vertical motion of a particle with air resistance prescribe the rate of change of velocity in terms of velocity:

$$\frac{dv}{dt} = g - \frac{k}{m}v \quad \cdots \quad (3)$$

Or depending on the model used for the resistance force,

$$\frac{dv}{dt} = g - \frac{k}{m}v^2 \quad \cdots \quad (4)$$

Equations (3) and (4) are first order differential equations specifying the velocity as a function of time. Equation (3) is a linear first order differential equation since v and $\frac{dv}{dt}$ appear in the equation without products such as $v \times v$, $\frac{dv}{dt} \times \frac{dv}{dt}$ or $\frac{dv}{dt} \times v$. Equation (4) is nonlinear because v^2 appears in the equation. These first order differential equations (3) and (4) are also in a standard form, namely,

$$\frac{dy}{dx} = f(x)g(y) \quad \cdots \quad (5)$$

The key point being the derivative can be expressed as the product of two function where one function expresses a relationship between the dependent variable y while the other only involves the independent variable x. For equation (4) $f(t) = 1$ and $g(v) = g - \frac{k}{m}v^2$. The solution for the standard form (5) is obtained by assuming

$$\int \frac{1}{g(y)} dy = \int f(x)\, dx + C \quad \cdots \quad (6)$$

The solution relies on the separation of the variables. For Equation (3), $f(t) = 1$ and $g(v) = g - \frac{k}{m}v$, therefore the solution can be obtained as follows:

$$\int \frac{1}{g - \frac{k}{m}v} dv = \int 1\, dt + C$$

$$-\frac{m}{k} ln\left(\left|g - \frac{k}{m}v\right|\right) = t + C \Rightarrow\ g - \frac{k}{m}v = e^{-\frac{k}{m}(t+C)} \ \Rightarrow v = \frac{mg}{k} - e^{-\frac{k}{m}C}e^{-\frac{k}{m}t}$$

If the is particle initially released from rest, then $v = 0$ when $t = 0$, therefore $e^{-\frac{k}{m}C} = \frac{mg}{k}$, hence

$$v(t) = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

The same procedure could be used to find a solution for the nonlinear differential equation (4). Equation (3) represents a first order linear differential equation for which two standard forms can apply. In addition to being open to direct integration using (5) and (6), Equation (3) is of the form:

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \cdots \quad (7)$$

Differential equations of the form (7) can be solved by determining a, so called, integrating factor $f(x)$ such that the differential equation can be reduced to an equivalent equation:

$$\frac{du}{dx} = g(x) \quad \cdots \quad (8)$$

If $u = f(x)y$, then by the rule for differentiating products

$$\frac{du}{dx} = \frac{d[f(x)y]}{dx} = y\frac{df(x)}{dx} + f(x)\frac{dy}{dx}$$

If Equation (7) is multiplied throughout by the integrating factor $f(x)$

$$f(x)\frac{dy}{dx} + f(x)P(x)y = f(x)Q(x) \quad \cdots \quad (9)$$

Equation (9) will reduce to Equation (8) provided

$$f(x)\frac{dy}{dx} + f(x)P(x)y = y\frac{df(x)}{dx} + f(x)\frac{dy}{dx} \quad \cdots \quad (10)$$

And

$$g(x) = f(x)Q(x)$$

Equation (10) is valid provided

$$f(x)P(x)y = y\frac{df(x)}{dx}$$

Or

$$\frac{df}{dx} = P(x)f$$

Applying the solution based on separation of variables yields

$$ln|f| = \int P(x)\, dx \Rightarrow f(x) = e^{\int P(x)\, dx}$$

Equation (9) can now be written in the form

$$e^{\int P(x)\, dx}\frac{dy}{dx} + e^{\int P(x)\, dx}P(x)y = e^{\int P(x)\, dx}Q(x) \quad \cdots \quad (11)$$

Therefore if $u = e^{\int P(x)\, dx}y$ is used in Equation (8), an equivalent differential equation to Equation (11) is obtained as follows

$$\frac{d\left[e^{\int P(x)\, dx}y\right]}{dx} = e^{\int P(x)\, dx}Q(x)$$

$$\Rightarrow e^{\int P(x)\, dx}y = \int e^{\int P(x)\, dx}Q(x)\ dx + C$$

Since Equation (3) can be written in the standard form defined by Equation (7), namely,

$$\frac{dv}{dt} + \frac{k}{m}v = g$$

We can therefore identify the following functions $P(t) = \frac{k}{m}$ and $Q(t) = g$, therefore the solution requires an integrating factor of $e^{\int \frac{k}{m} dt} = e^{\frac{k}{m}t}$, therefore

$$e^{\frac{k}{m}t} v = \int e^{\frac{k}{m}t} g\, dt + C$$

$$e^{\frac{k}{m}t} v = \frac{mg}{k} e^{\frac{k}{m}t} + C \Rightarrow v = \frac{mg}{k} + Ce^{-\frac{k}{m}t}$$

Applying the same initial conditions as before, namely, $v = 0$ when $t = 0$ yields $C = -\frac{mg}{k}$ resulting in the same answer as before

$$v(t) = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

Two different methods applied to a single problem leading to the same conclusion provide a sense of reassurance. An alternative to explicitly solving a differential equation is to calculate the solution using numerical methods. It is important to realise, however, that even when an expression or a numerical solution is produced, there is the possibility an assumption used in the solution is invalid and therefore the solution is only valid for a limited range of the independent variable. An equation of the form

$$t\frac{dv}{dt} - v = t \qquad t > 0$$

requires the condition $t > 0$ since the solution involves $\frac{1}{t}$. The importance of such restrictions can be nicely illustrated by the follow sequence of algebraic steps applied to any number a leading to a contradiction.

$$a^2 - a^2 = (a + a)(a - a)$$

$$\Rightarrow a(a - a) = (a + a)(a - a)$$

So far so good, but attempting to divide by $(a - a)$ leads to

$$\nRightarrow a = (a + a)$$

$$\Rightarrow a = 2a \Rightarrow 1 = 2?$$

In terms of manipulation of numbers, these steps appear fine but for the step in which $a - a = 0$ is eliminated. Dividing by zero is clearly shown to produce an incorrect answer. Differential equations may have conditions leading to similar issues, but for now it is sufficient to understand the solution techniques for differential equations and defer these problematic considerations for those studying mathematics at a higher level than this text.

Second Order Linear Differential Equations with Constant Coefficients

Dynamics problems involving Newton's second law of motion often involve second order linear differential equations as illustrated in the derivation of Equation (1) for a particle attached to a light spring. For an understanding of simple harmonic motion it is sufficient to investigate the solution of differential equations with constant coefficients:

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t) \quad \cdots \quad (12)$$

That is, equations of the form (12) for which a, b and c are all constant.

The equation of motion for a particle attached to a light spring is of the form (12)

$$m\frac{d^2x}{dt^2} + \frac{\lambda}{l}x = 0 \quad \cdots \quad (13)$$

where $a = m$, $b = 0$, $c = \frac{\lambda}{l}$ and $f(t) = 0$.

Apart from being important mathematical methods for mechanics in their own right, solutions of first order differential equations play a role in solving equations of the form (12). Before writing down the solution for Equation (12), first the solution for the equation

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0 \quad \cdots \quad (14)$$

must be established.

While $\frac{dx}{dt}$ is a function obtained from the function $x(t)$, the act of differentiating $x(t)$ could be defined in the sense that

$$\frac{dx}{dt} = \left(\frac{d}{dt}\right)x$$

Similarly, the second derivative of $x(t)$ might be expressed as

$$\frac{d^2x}{dt^2} = \left(\frac{d}{dt}\right)\left(\frac{d}{dt}\right)x$$

Using these alternative forms for the first and second derivative of $x(t)$ Equation (14) could be expressed as

$$\left(a\left(\frac{d}{dt}\right)\left(\frac{d}{dt}\right) + b\frac{d}{dt} + c\right)x = 0 \quad \cdots \quad (15)$$

It might seem reasonable to think of these operations expressed by Equation (15) in an equivalent form using the analogy for factorising a quadratic equation

$$am^2 + bm + c = 0$$

as

$$(m - \alpha_1)(m - \alpha_2) = 0$$

$$\left(\left(\frac{d}{dt}\right) - \alpha_1\right)\left(\left(\frac{d}{dt}\right) - \alpha_2\right)x = 0 \quad \cdots \quad (16)$$

If Equations (15) and (16) are equivalent, then the solution $x(t)$ might reasonably be expected to be obtained from the first order differential equation

$$\left(\left(\frac{d}{dt}\right) - \alpha_2\right)x = 0 \; or \; \frac{dx}{dt} - \alpha_2 x = 0 \quad \cdots \quad (17)$$

Applying separation of variables

$$\int \frac{1}{x} dx = \int \alpha_2 dt + C \Rightarrow ln|x| = \alpha_2 t + C \Rightarrow x(t) = e^{(\alpha_2 t + C)} = Be^{\alpha_2 t}$$

Thus a solution to Equation (16) obtain from the methods above is $x(t) = Be^{\alpha_2 t}$.

Since the roots for the quadratic polynomial are also interchangeable when Equation (17) was chosen, it might also be reasonable to assume $x(t) = Ae^{\alpha_1 t}$ is also a function which satisfies Equation (14).

Since

$$x(t) = Ae^{\alpha_1 t}$$

$$\Rightarrow \frac{dx}{dt} = A\alpha_1 e^{\alpha_1 t}$$

$$\Rightarrow \frac{d^2x}{dt^2} = A{\alpha_1}^2 e^{\alpha_1 t}$$

Therefore substituting into Equation (14)

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = aA{\alpha_1}^2 e^{\alpha_1 t} + bA\alpha_1 e^{\alpha_1 t} + cAe^{\alpha_1 t} = Ae^{\alpha_1 t}(a{\alpha_1}^2 + b\alpha_1 + c)$$

And since the value $\alpha_1$is a root of $am^2 + bm + c = 0 \Rightarrow a{\alpha_1}^2 + b\alpha_1 + c = 0$, hence $x(t) = Ae^{\alpha_1 t}$ is a solution of the differential equation (14). Similarly, $x(t) = Be^{\alpha_2 t}$ must be a solution and since $0 + 0 = 0$,

$$x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t} \quad \cdots \quad (18)$$

is also a solution of the differential equation (14).

Equation (18) is consistent with the previous discussion about potential loss of information resulting from differentiating a function, namely, the second derivative of a function potentially needs two constants of integration to allow for a class of functions all of which have the same second derivative. The introduction of two constants in the solution serves to introduce the necessary generality needed to accommodate the range of functions $x(t)$ satisfying Equation (14).

Repeated Root for $am^2 + bm + c = 0$

The generality of the solution (18) runs into problems if the quadratic equation $am^2 + bm + c = 0$ has repeated roots α, in which case Equation (18) reduces to

$$x(t) = Ae^{\alpha t} + Be^{\alpha t} = (A + B)e^{\alpha t} = Ce^{\alpha t}$$

Namely only a single constant and function appear in the solution. It becomes necessary to look for a further solution before all the possible solutions to the differential equation are obtained. It can be shown that if $x(t) = Ae^{\alpha t}$ is a solution of (14), then $x(t) = Bte^{\alpha t}$ is also a solution of (14). The fact that a second solution is required and the method for constructing the second solution are both consequences of theory beyond the scope of this text, so simply showing that $x(t) = Bte^{\alpha t}$ is a solution of (14) will suffice.

$$x(t) = Bte^{\alpha t}$$

$$\Longrightarrow \frac{dx}{dt} = Be^{\alpha t} + Bt\alpha e^{\alpha t} \Longrightarrow \frac{d^2x}{dt^2} = B\alpha e^{\alpha t} + B\alpha e^{\alpha t} + Bt\alpha^2 e^{\alpha t} = 2B\alpha e^{\alpha t} + Bt\alpha^2 e^{\alpha t}$$

Substituting into the left-hand side of (14)

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = a(2B\alpha e^{\alpha t} + Bt\alpha^2 e^{\alpha t}) + b(Be^{\alpha t} + Bt\alpha e^{\alpha t}) + cBte^{\alpha t}$$

$$= Bte^{\alpha t}(a\alpha^2 + b\alpha + c) + \ Be^{\alpha t}(2\alpha a + b)$$

If α is a repeated root of $am^2 + bm + c = 0$ then

$$b^2 - 4ac = 0 \Longrightarrow \alpha = -\frac{b}{2a} \text{ so } a\alpha^2 + b\alpha + c = 0 \text{ and } 2\alpha a + b = 0$$

$$\therefore \quad a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = Bte^{\alpha t}(a\alpha^2 + b\alpha + c) + \ Be^{\alpha t}(2\alpha a + b) = 0$$

For repeated roots of the auxiliary equation $am^2 + bm + c = 0$, the general solution of (14) is

$$x(t) = (A + Bt)e^{\alpha t}$$

Complex Roots for $am^2 + bm + c = 0$

The motivation for considering differential equations was the equation of motion for a particle attached to a light spring. The resulting differential equation is written in the form of a second order differential equation with constant coefficients:

$$\frac{d^2x}{dt^2} + cx = 0 \quad \cdots \quad (19)$$

The auxiliary equation is therefore

$$m^2 + c = 0 \; where \; c > 0$$

This quadratic equation has no real roots, however the complex roots are $\alpha_1 = i\sqrt{c} = i\omega$ and $\alpha_2 = -i\sqrt{c} = -i\omega$, where $i = \sqrt{-1}$ and $\omega^2 = c$. The solution (18) still applies in the sense

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t}$$

Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, the complex solution to Equation (19) is

$$x(t) = A(\cos\omega t + i\sin\omega t) + B(\cos\omega t - i\sin\omega t)$$

$$x(t) = (A + B)\cos\omega t + i(A - B)\sin\omega t \quad \cdots \quad (20)$$

While expressed as a complex valued function of a real variable, the Equation (20) suggests $x(t) = \cos \omega t$ and $x(t) = \sin \omega t$ are solutions of Equation (19).

First consider $x(t) = \cos \omega t$

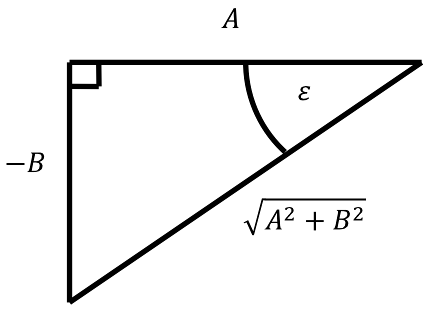
$$\Rightarrow \frac{dx}{dt} = -\omega \sin \omega t$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\omega^2 \cos \omega t = -\omega^2 x$$

Therefore $x(t) = \cos \omega t$ is indeed a solution of (19). Similarly $x(t) = \sin \omega t$ is another solution. The real valued general solution of (19) is therefore of the form

$$x(t) = A \cos \omega t + B \sin \omega t \quad \cdots \quad (21)$$

Defining the alternative constants a and ε as follows:



$$a = \sqrt{A^2 + B^2}$$

$$\cos \varepsilon = \frac{A}{\sqrt{A^2 + B^2}}$$

$$\sin \varepsilon = \frac{-B}{\sqrt{A^2 + B^2}}$$

The solution to equation (19) can be written as follows:

$$x(t) = a \cos \varepsilon \cos \omega t - a \sin \varepsilon \sin \omega t$$

$$x(t) = a\,(\cos \varepsilon \cos \omega t - \sin \varepsilon \sin \omega t)$$

Using $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ the solution can be expressed in the form

$$x(t) = a \cos(\omega t + \varepsilon) \quad \cdots \quad (22)$$

The solution (22) is an alternative formulation of solution (21), in which the constants a and ε can be interpreted as the amplitude or maximum displacement from the centre for the oscillation of a particle attached to a spring and as defining the initial displacement of the particle at the time $t = 0$. Equation (22) is the more common form used when analysing dynamics problems described as simple harmonic motion, of which a particle on a spring is one example of this type of motion.

More generally, the auxiliary equation $am^2 + bm + c = 0$ has complex roots of the form $\alpha_1 = p + iq$ and $\alpha_2 = p - iq$ whenever the $b^2 - 4ac < 0$ and $b \neq 0$. Under these circumstances the solution as prescribed by Equation (18) takes the form:

$$x(t) = Ae^{(p+iq)t} + Be^{(p-iq)t} \quad \Rightarrow \quad x(t) = e^{pt}\left(Ae^{iqt} + Be^{-iqt}\right)$$

Following a similar analysis used to obtain Equation (20) the complex valued solution is of the form

$$x(t) = e^{pt}((A+B)\cos qt + i(A-B)\sin qt)$$

Since the differential equation (14) has real coefficients and equates to zero, it might be reasonable to assume both real and imaginary part of the complex solution must be solutions of the differential equation (14). A solution of the form $x(t) = e^{pt}\sin qt$ is therefore a nature first choice to test by substitution into the differential equation.

$$x(t) = e^{pt}\sin qt$$

$$\Rightarrow \frac{dx}{dt} = pe^{pt}\sin qt + qe^{pt}\cos qt = e^{pt}(p\sin qt + q\cos qt)$$

$$\Rightarrow \frac{d^2x}{dt^2} = pe^{pt}(p\sin qt + q\cos qt) + e^{pt}(pq\cos qt - q^2\sin qt)$$

$$\Rightarrow \frac{d^2x}{dt^2} = e^{pt}(p^2\sin qt + 2pq\cos qt - q^2\sin qt)$$

Substituting into

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx$$
$$= ae^{pt}(p^2\sin qt + 2pq\cos qt - q^2\sin qt) + be^{pt}(p\sin qt + q\cos qt) + ce^{pt}\sin qt$$

Therefore,

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = e^{pt}[(a(p^2-q^2) + bp + c)\sin qt + (2ap+b)q\cos qt]$$

Since the complex roots of the auxiliary equation $am^2 + bm + c = 0$ are obtained from

$$\alpha = -\frac{b}{2a} \pm i\frac{\sqrt{4ac-b^2}}{2a}$$

it follows that $p = -\frac{b}{2a}$ and $q = \frac{\sqrt{4ac-b^2}}{2a}$, therefore

$$a(p^2-q^2) + bp + c = a\left(\left(-\frac{b}{2a}\right)^2 - \left(\frac{\sqrt{4ac-b^2}}{2a}\right)^2\right) + b\left(-\frac{b}{2a}\right) + c$$

$$\Rightarrow \quad a(p^2-q^2) + bp + c = \frac{b^2}{4a} - \frac{4ac}{4a} + \frac{b^2}{4a} - \frac{b^2}{2a} + c = 0$$

Similarly

$$2ap + b = 2a\left(-\frac{b}{2a}\right) + b = 0$$

Thus, $x(t) = e^{pt}\sin qt$ is a solution of $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$ whenever the auxiliary equation $am^2 + bm + c = 0$ has complex roots $\alpha_1 = p + iq$ and $\alpha_2 = p - iq$, with $q \neq 0$.

A similar argument shows $x(t) = e^{pt}\cos qt$ is also a solution of (14) and therefore a real valued solution of $a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$ can be expressed in the form:

$$x(t) = e^{pt}(A\cos qt + B\sin qt) \quad \cdots \quad (23)$$

To solve a second-order homogeneous linear differential equation of the form:

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

Determine the root α_1 and α_2 of the auxiliary equation

$$am^2 + bm + c = 0$$

The general solution for the differential equation is then one of the following three options:

Roots of Auxiliary Equation α_1 and α_2	General Solution
Real roots: $\boldsymbol{\alpha_1 \neq \alpha_2}$	$x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$
Real roots: $\boldsymbol{\alpha_1 = \alpha_2 = \alpha}$	$x(t) = (A + Bt)e^{\alpha t}$
Complex Roots: $\boldsymbol{\alpha_1 = p + iq \text{ and } \alpha_1 = p - iq}$	$x(t) = e^{pt}(A\cos qt + B\sin qt)$

There are problems in mechanics for which the homogeneous differential equation is replaced by an equation of the form:

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t) \quad \cdots \quad (24)$$

For example, the motion of a particle of mass m attached to a spring with a constant additional force in the direction of the x-axis given by mX results in the equation of motion:

$$m\frac{d^2x}{dt^2} = -\frac{\lambda}{l}x + mX$$

Or

$$\frac{d^2x}{dt^2} + \frac{\lambda}{ml}x = X$$

The general solution for these types of problems reduces to three steps:

1. Find the general solution $x_c(t)$ for the complementary homogenous equation

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

2. Find any function $x_p(t)$ not part of the complementary solution which satisfies

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t)$$

3. Add the two solutions together to form the general solution for (24)

$$x(t) = x_c(t) + x_p(t)$$

The function referred to as $x_p(t)$ is known as a **particular integral**, while $x_c(t)$ is the **complementary function**. Since the complementary function is established using the table above, the problem is therefore to construct a particular integral for equations of the form (24).

Consider the equation of motion:

$$\frac{d^2x}{dt^2} + \frac{\lambda}{ml}x = X$$

If $\omega^2 = \frac{\lambda}{ml}$ then the solution for $\frac{d^2x}{dt^2} + \omega^2 x = 0$ has already been established to be

$$x_c(t) = a\cos(\ \omega t + \varepsilon)$$

The problem is to find the simplest function $x_p(t)$ such that

$$\frac{d^2x_p}{dt^2} + \omega^2 x_p = X$$

Since the function $x_p(t)$ appears on the left-hand-side if $x_p(t) = \beta$ is used as a solution where β is a constant, the first and second derivatives of $x_p(t)$ are zero and therefore

$$0 + \omega^2 c = X \Longrightarrow \beta = \frac{X}{\omega^2}$$

The general solution is therefore

$$x(t) = x_c(t) + x_p(t)$$

$$x(t) = a\cos(\ \omega t + \varepsilon) + \frac{X}{\omega^2}$$

For these specific types of second order differential equations it is possible to find many different particular integrals, however it can be shown that the general solution constructed from the complementary function and any one of these particular integrals result in the same answer when boundary conditions are applied of the form $x(t_0) = \alpha$ and $\left.\frac{dx}{dt}\right|_{t_0} = \beta$.

Methods for determining particular integral for differential equations of the form (24) are again beyond the scope of this text, so a limited number of special cases will be tabulated with their use illustrated by example.

Form for f(t)	Form for Particular Integral
$\boldsymbol{f(t) = a_0}$	$x_p(t) = b_0$
$\boldsymbol{f(t) = a_0 + a_1 t}$	$x_p(t) = b_0 + b_1 t$
$\boldsymbol{f(t) = a_0 + a_1 t + a_2 t^2}$	$x_p(t) = b_0 + b_1 t + b_2 t^2$
$\boldsymbol{f(t) = ae^{ct}}$	$x_p(t) = be^{ct}$
$\boldsymbol{f(t) = a_0 \cos ct + a_1 \sin ct}$	$x_p(t) = b_0 \cos ct + b_1 \sin ct$

To illustrate the use of these particular integrals, consider the problem of a particle attached to a spring, where instead of a constant force mX the disturbing force varies with time according to $mk \cos ct$. Newton's second law of motion yields the equation

$$m\frac{d^2x}{dt^2} = -\frac{\lambda}{l}x + mk\cos ct$$

Or if $\omega^2 = \frac{\lambda}{ml}$

$$\frac{d^2x}{dt^2} + \omega^2 x = k\cos ct \quad \cdots \quad (25)$$

Solving Equation (25) involves determining the complementary function $x_c(t)$ for the homogeneous equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

and finding an appropriate particular integral $x_p(t)$ for the function $f(t) = k\cos ct$. Since the form for the function $f(t)$ matches $f(t) = a_0 \cos ct + a_1 \sin ct$ where $a_1 = 0$, the particular integral must be of the form $x_p(t) = b_0 \cos ct + b_1 \sin ct$, where both b_0 and b_1 must be determined by substituting into the identity

$$\frac{d^2x_p}{dt^2} + \omega^2 x_p \equiv k\cos ct$$

Given the form $x_p(t) = b_0 \cos ct + b_1 \sin ct$

$$\frac{dx_p}{dt} = -cb_0 \sin ct + cb_1 \cos ct$$

And

$$\frac{d^2x_p}{dt^2} = -c^2 b_0 \cos ct - c^2 b_1 \sin ct = -c^2(b_0 \cos ct + b_1 \sin ct)$$

Therefore

$$\frac{d^2x_p}{dt^2} + \omega^2 x_p = -c^2(b_0 \cos ct + b_1 \sin ct) + \omega^2(b_0 \cos ct + b_1 \sin ct) \equiv k\cos ct$$

Equating coefficients for sine and cosine yields

$$-c^2 b_0 + \omega^2 b_0 = k \Longrightarrow\ b_0 = \frac{k}{\omega^2 - c^2}\ for\ c \neq \omega$$

$$-c^2 b_1 + \omega^2 b_1 = 0\ \Longrightarrow\ b_1 = 0$$

Therefore for $c \neq \omega$ the particular integral is

$$x_p(t) = \frac{k}{\omega^2 - c^2} \cos ct$$

And the general solution is

$$x(t) = x_c(t) + x_p(t)$$

$$x(t) = a \cos(\ \omega t + \varepsilon) + \frac{k}{\omega^2 - c^2} \cos ct \quad c \neq \omega$$

Example:

Given the boundary conditions $x = 0$ and $\frac{dx}{dt} = 4$ at $t = 0$ and the differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 4e^{-t}$$

Find x as a function of t.

Solution:

The first step is to calculate the general solution to the homogenous differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$$

It is important to determine the complementary function first since there is always the possibility the standard option for the particular integral corresponding to $f(t) = 4e^{-t}$ is included in the two functions used to construct the complementary function. Obtaining the complement function allows an informed decision to be made when selected the form for the particular integral.

The auxiliary equation corresponding to $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$ is

$$m^2 + 5m + 6 = 0 \Longrightarrow (m + 3)(m + 2) = 0$$

The complementary function is therefore constructed using the form for two distinct real roots:

$$x_c(t) = Ae^{-2t} + Be^{-3t}$$

Since $f(t) = 4e^{-t}$ cannot be constructed from $x_c(t)$ the particular integral can be chosen to be

$$x_p(t) = be^{-t} \Longrightarrow \frac{dx_p}{dt} = -be^{-t} \Longrightarrow \frac{d^2x}{dt^2} = be^{-t}$$

Substituting into

$$\frac{d^2x_p}{dt^2} + 5\frac{dx_p}{dt} + 6x_p = be^{-t} + 5(-be^{-t}) + 6be^{-t} = 2be^{-t}$$

The particular integral must satisfy the differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 4e^{-t}$$

Therefore

$$4e^{-t} \equiv 2be^{-t} \Longrightarrow 4 = 2b \Longrightarrow b = 2 \Longrightarrow x_p(t) = 2e^{-t}$$

The general solution for the differential equation is

$$x(t) = x_c(t) + x_p(t)$$

$$x(t) = 2e^{-t} + Ae^{-2t} + Be^{-3t}$$

Applying the boundary conditions $x = 0$ and $\frac{dx}{dt} = 4$ at $t = 0$

$$0 = x(0) = 2e^{0} + Ae^{-2(0)} + Be^{-3(0)} \Longrightarrow 2 + A + B = 0 \quad \cdots \quad (a)$$

$$\frac{dx}{dt} = -2e^{-t} - 2Ae^{-2t} - 3Be^{-3t}$$

Therefore $\frac{dx}{dt} = 4$ at $t = 0$ results in the equation for the constants A and B

$$4 = -2e^{(0)} - 2Ae^{-2(0)} - 3Be^{-3(0)} \Longrightarrow 6 + 2A + 3B = 0 \quad \cdots \quad (b)$$

Solving the simultaneous equations (a) and (b) yields $A = 0$ and $B = -2$. The boundary conditions result is the particular solution

$$x(t) = 2e^{-t} - 2e^{-3t}$$

Example:

A particle of mass m attached to a spring is subject to three forces:

i. A tension force $-m9x$
ii. A damping force proportional to the velocity $-m2\frac{dx}{dt}$
iii. A disturbing force $m3\sin 2t$

By applying Newton's second law of motion, express the displacement x in terms of time t as a differential equation and solve the differential equation for the general solution $x(t)$.

Solution:

Newton's second law of motion $F = ma$ allows these three forces to be combined in the form

$$m\frac{d^2x}{dt^2} = -m9x - m2\frac{dx}{dt} + m3\sin 2t$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 9x = 3\sin 2t \quad \cdots \quad (a)$$

Equation (a) is a second order linear differential equation with constant coefficients. The solution is therefore of the form $x(t) = complementary\ function + particular\ intergral.$

The complementary function is obtained from the general solution for the corresponding homogeneous equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 9x = 0 \quad \cdots \quad (b)$$

The auxiliary equation for Equation (b) is

$$m^2 + 2m + 9 = 0$$

Since $b^2 - 4ac = 2^2 - 4(1)(9) = -32 < 0$ the roots for the auxiliary equation are complex and therefore the complementary function is of the form

$$x_c(t) = e^{pt}(A\cos qt + B\sin qt)$$

Where $p = -\frac{b}{2a} = -\frac{2}{2(1)} = -1$ and $q = \frac{\sqrt{4ac-b^2}}{2a} = \frac{\sqrt{32}}{2(1)} = 2\sqrt{2}$.

$$x_c(t) = e^{-t}(A\cos 2\sqrt{2}t + B\sin 2\sqrt{2}t)$$

The particular integral $x_p(t)$ for the function $f(t) = 3\sin 2t$. Since the form for the function $f(t)$ matches $f(t) = a_0\cos ct + a_1\sin ct$ where $a_0 = 0$ and $c = 2$, the particular integral must be of the form $x_p(t) = b_0\cos 2t + b_1\sin 2t$, where both b_0 and b_1 must be determined by substituting into the identity

$$\frac{d^2x_p}{dt^2} + 2\frac{dx_p}{dt} + 9x_p \equiv 3\sin 2t$$

Given the form $x_p(t) = b_0\cos 2t + b_1\sin 2t$

$$\frac{dx_p}{dt} = -2b_0\sin 2t + 2b_1\cos 2t$$

And

$$\frac{d^2x_p}{dt^2} = -2^2b_0\cos 2t - 2^2b_1\sin 2t = -4(b_0\cos 2t + b_1\sin 2t)$$

Therefore

$$\frac{d^2x_p}{dt^2} + 2\frac{dx_p}{dt} + 9x_p = -4(b_0\cos 2t + b_1\sin 2t) +$$

$$2(-2b_0\sin 2t + 2b_1\cos 2t) + 9(b_0\cos 2t + b_1\sin 2t) \equiv 3\sin 2t + 0\cos 2t$$

Collecting coefficients of $\cos 2t$ and $\sin 2t$:

Coefficient of $\sin 2t$:

$$-4b_1 - 4b_0 + 9b_1 = 3 \Rightarrow 5b_1 - 4b_0 = 3$$

Coefficient of $\cos 2t$:

$$-4b_0 + 4b_1 + 9b_0 = 0 \Rightarrow b_1 = -\frac{5}{4}b_0$$

$$5\left(-\frac{5}{4}b_0\right) - 4b_0 = 3 \Rightarrow b_0 = -\frac{12}{41} \quad and \quad b_1 = \frac{15}{41}$$

The particular integral for Equation (a) is therefore

$$x_p(t) = -\frac{12}{41}\cos 2t + \frac{15}{41}\sin 2t$$

With general solution

$$x(t) = x_c(t) + x_p(t)$$

$$x(t) = e^{-t}(A\cos 2\sqrt{2}t + B\sin 2\sqrt{2}t) + -\frac{12}{41}\cos 2t + \frac{15}{41}\sin 2t$$

Reduction of Differential Equations to Standard Forms by Substitution

The discussions above are concerned with finding solutions to a select group of differential equations appearing in standard forms, namely,

1. first order differential equations where the variables can be separated to allow direct integration,
2. first order linear differential equations by determining an integrating factor and
3. second order linear differential equations with constant coefficients.

These standard forms can also be useful for problems where a change of variable transforms a differential equation from one form to a standard form for which a solution can be found.

By way of example, consider the second order differential equation for a particle attached to a light spring.

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Rather than treating the problem as a second order differential equation, using

$$\frac{dv}{dt} = \frac{d^2x}{dt^2} \ and \ \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$$

Therefore

$\frac{d^2x}{dt^2} + \omega^2 x = 0$ is can be expressed as a first order differential equation involving velocity v and displacement x:

$$v\frac{dv}{dx} + \omega^2 x = 0$$

Substituting $u = \frac{1}{2}v^2$ implies $\frac{du}{dx} = v\frac{dv}{dx}$, therefore the equation is transformed to a first order linear differential equation

$$\frac{du}{dx} + \omega^2 x = 0$$

Using direct integration

$$u = -\omega^2 \int x\, dx + \ c \ \Rightarrow u = -\omega^2 \frac{x^2}{2} + c$$

Since $u = \frac{1}{2}v^2$

$$\frac{1}{2}v^2 = -\omega^2 \frac{x^2}{2} + c \Rightarrow v^2 = 2c - \omega^2 x^2$$

If the constant of integration is rewritten in the for $2c = \omega^2 a^2$ (imposing $2c - \omega^2 x^2 \geq 0$ which is require by $v^2 \geq 0$)

$$v = \pm\omega(a^2 - x^2)^{\frac{1}{2}} \quad \cdots \quad (26)$$

Equation (26) relates velocity to displacement and since $v = \frac{dx}{dt}$ is also a first order differential equation for $x(t)$.

$$\frac{dx}{dt} = \pm\omega(a^2 - x^2)^{\frac{1}{2}}$$

Using separation of variables

$$\int \frac{1}{(a^2 - x^2)^{\frac{1}{2}}} dx = \pm\, \omega \int dt + C$$

$$\Rightarrow \int \frac{1}{(a^2 - x^2)^{\frac{1}{2}}} dx = \pm\, \omega t + C$$

Making the substitution $x = a\cos\theta \Rightarrow dx = -a\sin\theta\, d\theta$

$$\int \frac{1}{(a^2 - x^2)^{\frac{1}{2}}} dx = -\int \frac{a\sin\theta}{a(1 - \cos^2\theta)^{\frac{1}{2}}} d\theta = -\int \frac{a\sin\theta}{a\sin\theta} d\theta = -\int d\theta$$

$$\therefore \quad \int \frac{1}{(a^2 - x^2)^{\frac{1}{2}}} dx = -\,\theta$$

Hence

$$-\theta = \pm\omega t + C$$

Since $x = a\cos\theta$,

$$x(t) = a\cos(\mp\omega t - C)$$

These two solutions are of the form

$$x(t) = a\cos(\omega t + \varepsilon) \quad \cdots \quad (27)$$

That is, the same solution for the original differential equation is recovered by direct integration as by applying the theory for the second order linear differential equation to the displacement as a function of time. Equation (22) and Equation (27) are identical.

Simple Harmonic Motion

While the introductory problems in mechanics involving the motion of a particle are often concerned with moving a particle from one place to another, there is an important class of problems where a particle goes through a motion, but at some point in the trajectory the particle returns to the initial position. An obvious example of repetitious motion is a Formula 1 race car which must execute a sequence of laps of a race circuit. Other examples might be the hands of an analogue clock or the vibrations in a tuning fork. The key characteristic for all these motions is that after a time period, the particle or particles retraces over ground previously encountered.

While periodic motion is often complex in nature, many problems can be reduced by approximation to a more simple form known as Simple Harmonic Motion (SHM). An example of such an approximation is a simple pendulum, where for small oscillations the motion can be approximated to simple harmonic motion.

Simple Harmonic Motion is an oscillation of a particle in a straight line. The motion is characterised by a centre of oscillation, acceleration for the particle which is always directed towards the centre of oscillation, and the acceleration is proportional to the displacement of the particle from the centre of oscillation. These statements are encapsulated in the differential equation

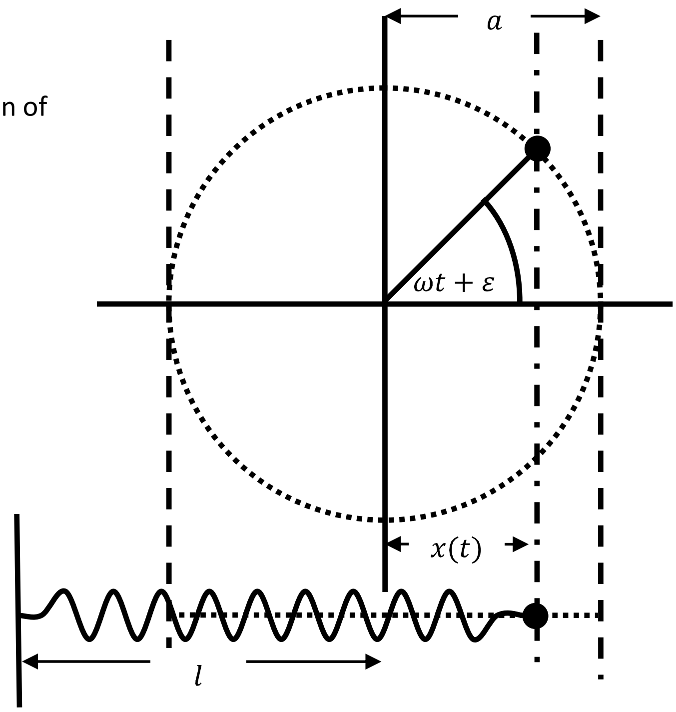
Simple Harmonic Motion

For constant ω the equation of motion

$$\ddot{x} + \omega^2 x = 0$$

has solution

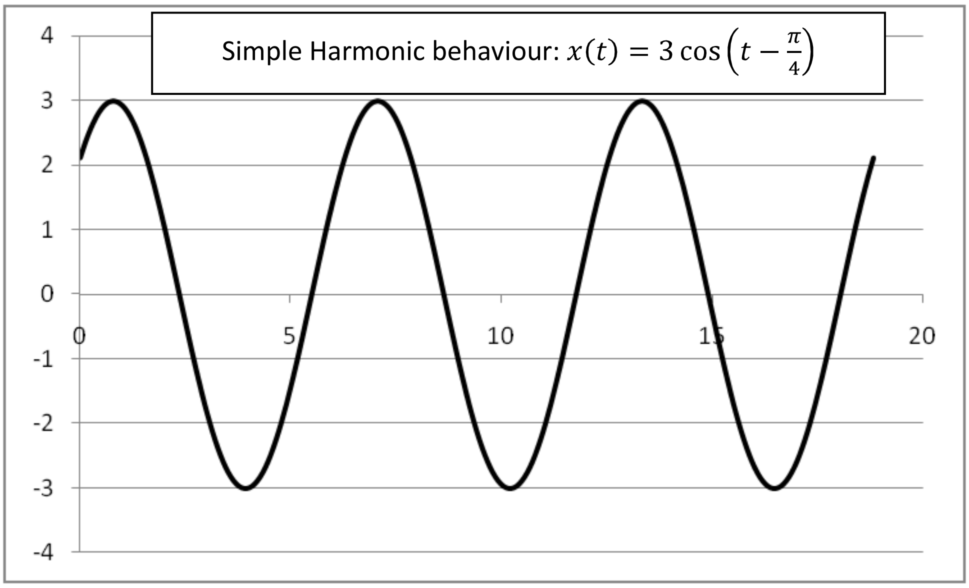
$$x(t) = a\cos(\omega t + \varepsilon)$$



Such a linear motion is precisely the motion of a particle of mass m attached to a spring of natural length l when moving on a smooth horizontal surface after being displaced a distance a from the natural length of the spring before being released.

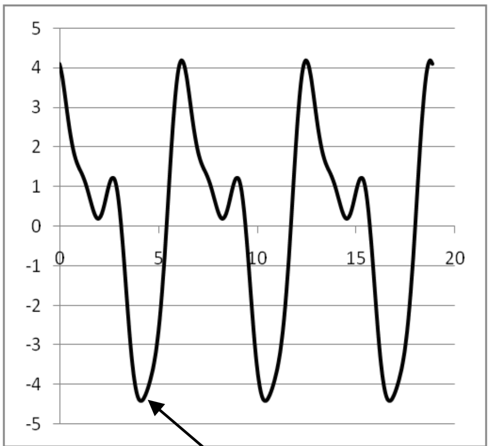
When approximating a motion as simple harmonic, the problem is reduced to that of a straight line trajectory for a particle corresponding to the x-coordinate of an equivalent particle moving in a circle of radius a with a constant speed. This motion of a particle in a circle provides a geometric perspective for simple harmonic motion expressed in the solution $x(t) = a\cos(\omega t + \varepsilon)$.

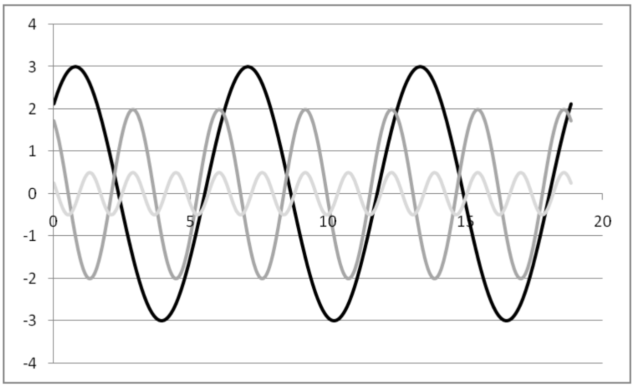
The trajectory for a particle undergoing Simple Harmonic Motion is described by a *simple* sine or cosine functions in terms of time, hence the name for the motion.



More complex oscillation can be analysed in terms of combinations of sine and cosine functions, so understanding the more fundamental problem of simple harmonic motion provides the basis for understanding these more general problems.

More complicated periodic motion can be recreated by combining these three sinusoidal motions representing simple harmonic oscillations. For example the motion of a piano string and be synthetically modelled from a number of sinusoidal motions.





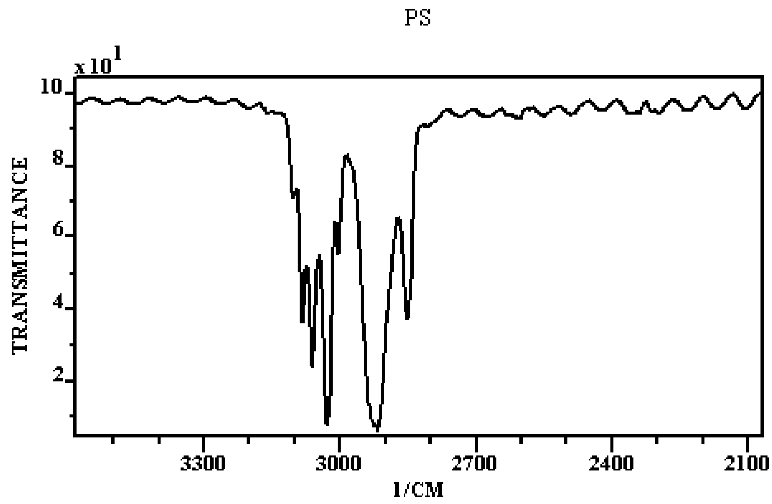
$$x_{SHM1}(t) = 3\cos\left(t - \frac{\pi}{4}\right)$$

$$x_{SHM2}(t) = 2\cos\left(2t + \frac{\pi}{6}\right)$$

$$x_{SHM3}(t) = \frac{1}{2}\cos\left(4t + \frac{\pi}{3}\right)$$

$$x(t) = x_{SHM1}(t) + x_{SHM2}(t) + x_{SHM3}(t)$$

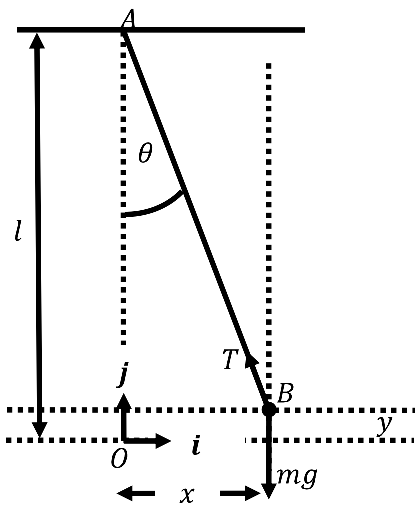
A technologically significant problem is that of interpreting infrared spectra, which can be understood in terms of oscillations associated with molecular bonds. A molecular bond is modelled as springs connecting two masses, hence the relevance of SHM. Infrared spectra are used to characterise materials for medical science and other key areas of technology. The following Fourier Transform Infrared (FTIR) spectrum illustrates numerous oscillations in intensity which can be traced back to vibrations associated with carbon-hydrogen bonds in polystyrene (PS).



FTIR Spectra from polystyrene.

The Simple Pendulum

A simple pendulum consists of a particle of mass m attached to one end of a light inextensible string of length l where the other end of the string is attached to a fixed point. The particle when at rest hangs vertically below the fixed point. The particle and string when displaced from the equilibrium position oscillate in a circular arc in the same vertical plane. This physical description suggests the particle moves in 2D and therefore the motion will not behave like simple harmonic motion. The value in studying the simple pendulum lies in observing the types of approximation and restrictions to the motion of the particle that allow a description in terms of simple harmonic motion.



A diagram for the simple pendulum showing the forces acting on the particle of mass m helps to write down the equations of motion $\boldsymbol{F} = m\boldsymbol{a}$ using the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$, to express the displacement from the origin O in terms of the tension T exerted by the inextensible string and weight mg:

$$\begin{pmatrix} -T\sin\theta \\ T\cos\theta - mg \end{pmatrix} = m\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$$

$$\cos\theta = \frac{l-y}{l} \text{ and } \sin\theta = \frac{x}{l}$$

In general, these equations for the simple pendulum do not match the equation for simple harmonic motion $\ddot{x} = -\omega^2 x$, however if the length of the string l is large compared with the vertical displacement y then $\frac{y}{l} \approx 0$, therefore

$$\cos\theta = \frac{l-y}{l} = 1 - \frac{y}{l} \approx 1$$

The assumption that l is large compared with y also suggests that the acceleration in the y direction is small too, therefore the $\boldsymbol{j}$ component for the equations of motion yields

$$T\cos\theta - mg = m\ddot{y} \Longrightarrow T - mg \approx 0$$

Applying these approximation to the equation of motion for the x direction

$$m\ddot{x} = -T\sin\theta$$

then becomes on replacing $T = mg$ and $\sin\theta = \frac{x}{l}$

$$m\ddot{x} = -m\frac{g}{l}x \Longrightarrow \ddot{x} = -\omega^2 x \Longrightarrow SHM$$

where $\omega = \sqrt{\frac{g}{l}}$

Thus, the motion of a simple pendulum for which the length of the string is large compared to the vertical displacement of the mass reduces under approximation to simple harmonic motion.

Note the condition that l is large compared with y is equivalent to stating the maximum angle θ for the oscillations is small. Also, the assertion that $\ddot{y} \approx 0$ is geometrically equivalent to observing for large l and small y the trajectory of the particle for small angles is almost without curve, that is, can be approximated by a straight line.

The reason for analysing a mechanical system such as the simple pendulum is to extract useful information. Historically, a pendulum offered a means of measuring time, the method being to count the number of complete oscillations. Once the motion of a pendulum is characterised in terms of simple harmonic motion, the mathematics of the solution $x(t) = a\cos(\omega t + \varepsilon)$ provides the means of calculating such useful parameters.

Solving Problems using Simple Harmonic Motion

Simple harmonic motion is referred to a periodic because after a time interval or period, the same trajectory for the particle begins afresh. This statement is mathematically described by the displacement $x(t)$ for the particle must be the same at two times t_1 and t_2:

$$a\cos(\omega t_2 + \varepsilon) = a\cos(\omega t_1 + \varepsilon)$$

$$\omega t_2 + \varepsilon = \omega t_1 + \varepsilon + n2\pi$$

$$\omega(t_2 - t_1) = n2\pi \Longrightarrow t_2 - t_1 = \frac{n2\pi}{\omega}$$

The shortest time $t_2 - t_1$ is called the period T and is therefore $T = \frac{2\pi}{\omega}$.

For a simple pendulum of length l, the time period is determined by $\omega = \sqrt{\frac{g}{l}}$ and is therefore

$$T = 2\pi\sqrt{\frac{l}{g}}$$

In general, if a problem can be expressed in the form of simple harmonic motion, that is, the equation of motion is of the form

$$\ddot{x} = -\omega^2 x$$

then the time for one complete oscillation is given by

$$T = \frac{2\pi}{\omega}$$

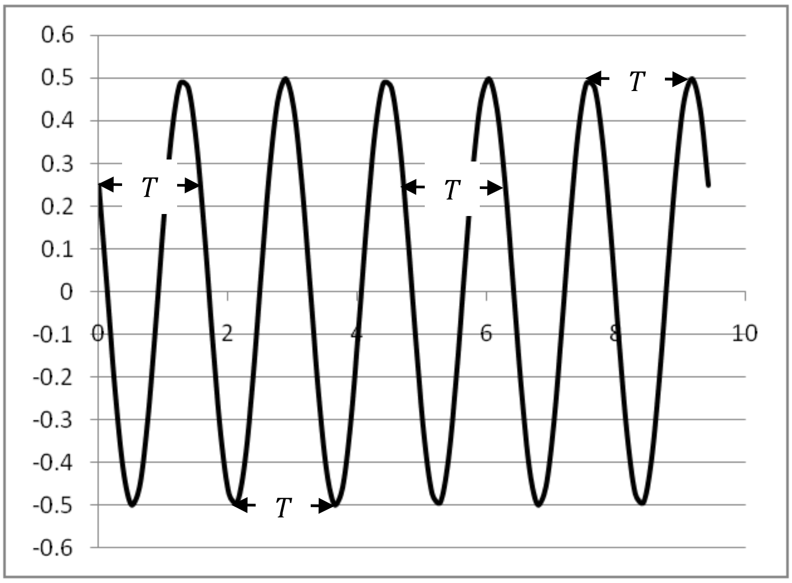
$$x_{SHM3}(t) = \frac{1}{2}\cos\left(4t + \frac{\pi}{3}\right)$$

Comparing solution to

$$x(t) = a\cos(\omega t + \varepsilon)$$

Implies $\omega = 4$

$$\therefore T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = 1.57$$



Note: the time period for a particle moving under simple harmonic motion is independent of the maximum displacement from the centre of oscillation a known as the amplitude. The velocity for the particle does depend on the amplitude.

Given the displacement for SHM,

$$x(t) = a\cos(\omega t + \varepsilon) \quad \cdots \quad (1)$$

$$v = \dot{x}(t) = -\omega a \sin\,(\omega t + \varepsilon) \quad \cdots \quad (2)$$

Eliminating t from these two equations by multiplying Equation (1) by ω before squaring and adding the resulting equations yields

$$v^2 + \omega^2 x^2 = (-\omega a \sin\,(\omega t + \varepsilon))^2 + \omega^2 a^2 \cos^2(\omega t + \varepsilon)$$

Using $\sin^2\theta + \cos^2\theta = 1$

$$v^2 + \omega^2 x^2 = (\sin^2(\omega t + \varepsilon) + \cos^2(\omega t + \varepsilon))\omega^2 a^2 = \omega^2 a^2$$

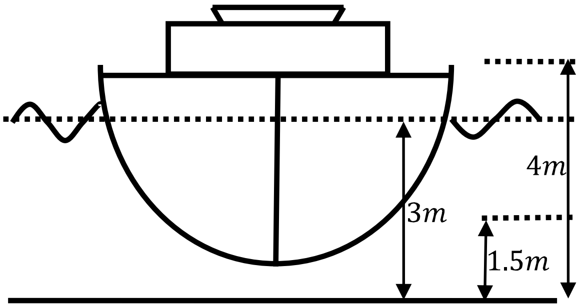
$$v^2 = \omega^2(a^2 - x^2) \quad \cdots \quad (3)$$

Equation (3) shows the velocity for a particle moving in SHM is a maximum when $x = 0$ and zero when the displacement of the particle is at either of the extreme positions from the centre of oscillation, namely, $x = \pm a$.

Example

The port at Teignmouth is in the Teign estuary. A sand bar at the mouth of the river Teign prevents ships from entering the port apart from when the tides raise the water level sufficiently to allow ships to pass over the sand bar and into the port.

The minimum and maximum water level due to tidal influences for a certain day is known to be $1.5\ m$ at $15{:}21$ hours and $4.0\ m$ at $21{:}46$ hours, respectively. By modelling the water level at the sand bar as varying according to simple harmonic motion, estimate the earliest time after low tide a ship requiring a depth of $3\ m$ can cross the sand bar and enter the port.



Solution

While the high and low water depth will vary on a daily basis, for the time interval between low water at $15{:}21$ and high water at $21{:}46$, the variation in tidal depths is sufficiently small to allow these variations to be approximated by a single sinusoidal function, hence the application of simple harmonic motion to the changes in water depth. Over a longer time interval, the approximation would breakdown, but for the problem as stated a reasonable estimate for the time at which a ship requiring $3\ m$ of water to pass safely can be calculated using simple harmonic motion.

This example states that simple harmonic motion can be used to approximate the water depth. The problem therefore does not involve showing that simple harmonic motion is appropriate, but simply requires the application of the solution to SHM to the conditions given in the question. The maximum and minimum depths are effectively boundary conditions for the SHM solution:

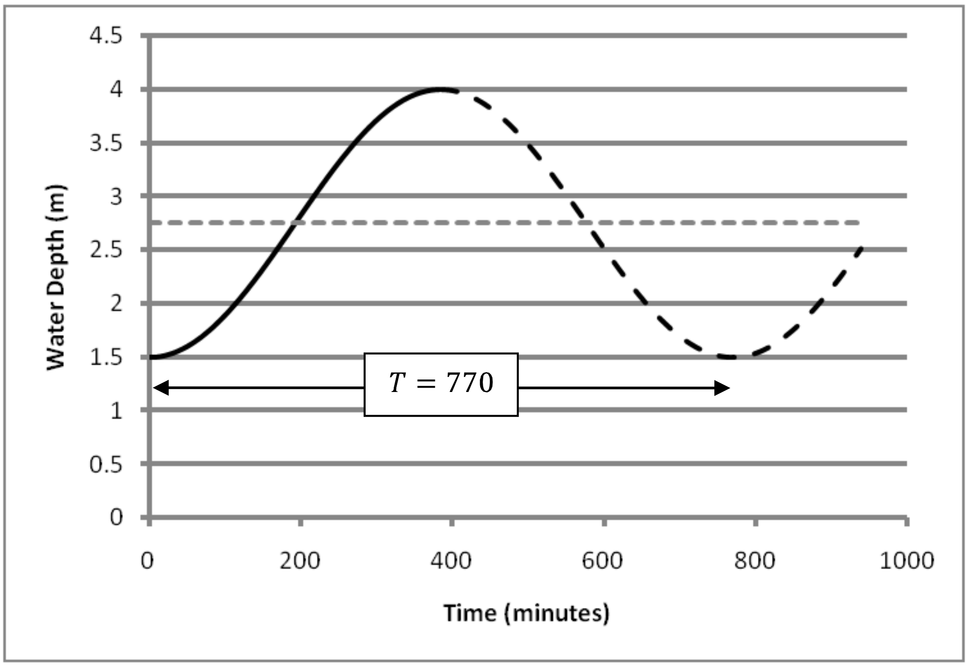
$$x_{SHM}(t) = a\cos(\omega t + \varepsilon)$$

where the simple harmonic oscillations occur about the mean depth for the water, namely,

$$1.5 + \frac{4 - 1.5}{2} = 2.75\ m$$

The actual depth of water is given by

$$x(t) = 2.75 + x_{SHM}(t)$$



The centre of oscillation for the SHM is 2.75 m, and the maximum displacement of the water from the mean depth is the amplitude for the SHM $a = \frac{4-1.5}{2} = 1.25\ m$.

These boundary conditions for the simple harmonic motions can be expressed in terms of displacement from the centre of oscillation for the water where $t = 0$ corresponds to low water and time is expressed in minutes. Since low water occurs at 15: 21, high water occurs 6 hrs 21 $minutes$ after low water or 385 $minutes$ after low water, thus

$$x_{SHM}(0) = -1.25\ m \text{ and } x_{SHM}(385) = 1.25\ m$$

A complete oscillation would cause the water to change for low water to high water and back to low water again, therefore the time period for the SHM will be twice the time to go from low water to high water. The time period T for the SHM is $T = 2 \times 385 = 770\ minutes$.

The relationship between ω and the time period is $\omega = \frac{2\pi}{T}$, therefore $\omega = \frac{2\pi}{770}$.

The SHM solutions will be completely determined once the phase shift ε is fixed. The phase shift establishes the displacement when $t = 0$ and since $x_{SHM}(0) = -1.25\ m$

$$1.25 \cos(0 + \varepsilon) = -1.25 \Longrightarrow \ \varepsilon = \pi$$

Therefore

$$x(t) = 2.75 + \ 1.25 \cos\left(\frac{2\pi}{770}t + \pi\right) = 2.75 - \ 1.25 \cos\left(\frac{2\pi}{770}t\right)$$

The time at which a ship requiring 3 m of water to pass over the sand bank is obtained from the equation

$$3 = 2.75 - 1.25\cos\left(\frac{2\pi}{770}t\right)$$

$$\Rightarrow \cos\left(\frac{2\pi}{770}t\right) = -\frac{0.25}{1.25} \Rightarrow \frac{2\pi}{770}t = 1.77 \Rightarrow t = 217.2\ minutes$$

Since time is measured from low tide at $15{:}\,21$, the ship must wait at least $3\ hours\ 37.2\ minutes$ before crossing the sand bar. The earliest time the ship should attempt to enter the river is estimated to be $18{:}\,59$.